Online Appendix to: Falsification-proof non-market allocation mechanisms

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Contents

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This Online Appendix supplements *Falsification-Proof Non-Market Allocation Mechanisms* by Eduardo Perez-Richet and Vasiliki Skreta.

Section S1 extends the baseline model to falsification cost functions with a fixed cost component, characterizing the optimal rule and establishing uniqueness under the (UID) assumption (Theorem S1.1). Section S2 introduces unobservable and observable group heterogeneity and exogenous quotas, deriving optimal within-group allocations (Theorem S2.1), characterizing the across-group optimum via a system of shadow prices (Theorem S2.2), and providing a finite algorithm (Algorithm 1) that computes a solution to the across problem; comparative statics and welfare results (Proposition S2.2) close the section. Section S3 contains additional comparative statics results for the optimal falsification-proof rule, covering shape under shifted linear costs (Proposition S3.1), the effect of returns to scale in falsification (Proposition S3.2), and the conditions on the score distribution that yield a uniformly higher allocation probability (Proposition S3.3 and Proposition S3.4). Finally, Section S4 justifies treating the continuum of agents as a single agent.

Throughout, we use the notation and assumptions of the main paper without restatement. Labels of the form (A.x), (B), (FPC), (UID), (UDD), (Mult), and (ZAS) refer to conditions defined in the main paper. Cross-references of the form *Appendix D, Proposition D.1* refer to the Appendix of the main paper.

S1 Fixed cost

We extend our analysis to a falsification cost function with a fixed cost in the (UID) case. The falsification cost from s to t is then given by

$$\bar{c}(t|s) = k\, \mathbb{1}_{t\neq s} + c(t|s),$$

where c is a cost function that satisfies (UID) and our standard assumptions. The falsification-proofness condition becomes:

$$\alpha(t) - \alpha(s) \leq k + c(t|s), \forall t \neq s. \tag{FPCfix}$$

We show that the optimal falsification-proof rule has the same structure as the one without fixed cost, with identical derivatives on the eligible and ineligible sides of the growth interval but a jump of size k at the eligibility threshold.

Monotonicity. We can adapt the proof of Lemma 2 and show that any falsification-proof rule is dominated by a monotonic falsification-proof rule, so there is no loss of generality in restricting attention to monotonic allocation rules. Monotonicity implies that allocation rules have at most countably many jump discontinuities (removable discontinuities can be removed without changing the allocative surplus, so we can assume them away). Furthermore, (FPCfix) implies that jumps are of size at most k. We can also make α right-continuous without affecting allocative surplus, so we can consider α as a right-continuous monotonic function bounded by $[0, 1]$.

We can then integrate by parts the objective function of the designer and rewrite it as the following Stieltjes integral

$$\int_S \mathcal{W}(x) d\alpha(x),$$

while optimally choosing $\alpha(\underline{s}) = 0$ under low-score priority and $\alpha(\overline{s}) = 1$ under high-score priority.

The differential program becomes

$$\max_{\alpha} \int_S \mathcal{W}(x) d\alpha(x)$$
$$\text{s.t. (FPCfix)}$$

Lagrangian sufficiency. Since α is a cdf, it satisfies the constraint $\int_S d\alpha(x) \leq 1$. Putting a Lagrange multiplier ν on this constraint yields the Lagrangian

$$\mathcal{L}(\alpha, \nu) = \int_S \{\mathcal{W}(x) - \nu\} d\alpha(x) + \nu.$$

It is clear that any α that puts mass outside of the interval $[s_*(\nu), s^*(\nu)]$ is strictly dominated. We prove the same Lagrangian sufficiency result as in the paper with the cost function $\bar{c}$. There is, however, an important difference in the proof. In the paper, the continuity property (ii) is immediate since any allocation rule satisfying (FPC) must be continuous. With the fixed cost, (FPCfix) no longer implies (ii). Instead, we show that for a rule α to maximize the Lagrangian $\mathcal{L}(\alpha, \nu)$ it is necessary to satisfy continuity and exhibit no jump at the extremities of the interval $\big[s_*(\nu), s^*(\nu)\big]$. This proof relies on (UID).

Lemma S1.1. *Assume* (UID)*. An allocation rule α solves* (P) *if and only if it satisfies* (PC)*,* (MON) *and* (Init)*, and there exists a Lagrange multiplier $\nu \geq 0$ such that:*

(i) ***Growth interval:*** *$\alpha(s) = \alpha(\underline{s})$ for every $s < s_*(\nu)$, and $\alpha(s) = \alpha(\overline{s})$ for every $s > s^*(\nu)$;*

(ii) ***Continuity:*** *α is continuous at $s_*(\nu)$ and at $s^*(\nu)$;*

(iii) ***Relaxed*** (FPCfix)***:*** *$\alpha(t) - \alpha(s) \leq \bar{c}(t|s)$ for every $s_*(\nu) \leq s < t \leq s^*(\nu)$;*

(iv) ***Complementary slackness on*** (PC)***:*** *If $\bar{w} \leq \hat{w}$, $\nu\big(1 - \alpha(\overline{s})\big) = 0$; if $\bar{w} \geq \hat{w}$, $\nu\alpha(\underline{s}) = 0$;*

(v) ***Optimality for the simplified program:*** *For every allocation rule $\hat{\alpha}$ that satisfies (iii),*

$$\int_{s_*(\nu)}^{s^*(\nu)} \{w(s) - \hat{w}\}\alpha(s)dF(s) \geq \int_{s_*(\nu)}^{s^*(\nu)} \{w(s) - \hat{w}\}\hat{\alpha}(s)dF(s).$$

Proof. We proceed in two steps as in the proof of Lemma A.1 in the paper. We only provide arguments for the parts of the proof that differ from that of Lemma A.1. Recall that we say an allocation rule α is *feasible*, and denote the set of such allocation rules by $\mathbb{A}$, if it satisfies (MON), (FPCfix), and (Init).

Step 1: *An allocation rule $\alpha \in \mathbb{A}$ solves* (P) *if and only if (A) α satisfies* (PC)*; and there exists $\nu \geq 0$ such that (B) $\nu = 0$ or $\overline{\alpha} - \underline{\alpha} = 1$, and (C) $\mathcal{L}(\alpha, \nu) \geq \mathcal{L}(\hat{\alpha}, \nu)$ for every allocation rule $\hat{\alpha} \in \mathbb{A}$.*

The proof of step 1 is identical to that in Lemma A.1 of the paper, so we proceed directly to the proof of step 2.

Step 2: *An allocation rule α and a scalar $\nu \geq 0$ satisfy* (MON)*,* (PC)*,* (Init)*, and (i)-(v) if and only if α and ν satisfy $\alpha \in \mathbb{A}$, (A), (B) and (C).*

$\boxed{\Rightarrow}$ The proof of this claim is identical to that in Lemma A.1, only replacing (FPC) by (FPCfix).

$\boxed{\Leftarrow}$ The difference with the proof of Lemma A.1 is that (FPCfix) does not imply the continuity property (ii). The rest of the proof is identical, so $\alpha \in \mathbb{A}$ directly implies (MON), (Init), and (iii). (A) directly implies (PC). (B) and (Init) imply (iv). Additionally, the proof that (C) implies (i) is identical. Finally, having proved (ii), the proof that (C) implies (v) is also identical. Hence, we only need to prove (ii).

We prove continuity by a different argument that relies on (UID): we show that (C) implies (ii). The proof is by contradiction. Suppose that $\alpha\big(s_*(\nu)^+\big) - \alpha\big(s_*(\nu)^-\big) = \delta >$

0. We choose $\tilde{s} \in \big(s_*(\nu), \hat{s}\big)$ sufficiently close to $s_*(\nu)$ so that $\eta = \delta - c\big(\tilde{s}|s_*(\nu)\big) > 0$. Then consider the function $\hat{\alpha}$ which is constant and equal to $\alpha\big(s_*(\nu)^+\big) - \eta > 0$ on $\big(s_*(\nu), \tilde{s}\big)$, and equal to α everywhere else. $\hat{\alpha}$ satisfies (MON) and (Init). We show that it also satisfies (FPCfix). The only case for which an argument is needed is for $s \in \big(s_*(\nu), \tilde{s}\big)$, and $t \in \big[\tilde{s}, s^*(\nu)\big]$. Then we have:

$$\begin{aligned}\hat{\alpha}(t) - \hat{\alpha}(s) &= \alpha(t) - \alpha\big(s_*(\nu)^+\big) + \eta \\ &= \alpha(t) - \alpha\big(s_*(\nu)^-\big) - c\big(\tilde{s}|s_*(\nu)\big) \\ &\leq k + c\big(t|s_*(\nu)\big) - c\big(\tilde{s}|s_*(\nu)\big) \\ &\leq k + c(t|\tilde{s}) \\ &\leq \bar{c}(t|s),\end{aligned}$$

where the first two equalities are by definition of $\hat{\alpha}$ and η, the first inequality is because α satisfies (FPCfix), the second inequality is by (UID), and the last inequality is by cost monotonicity for c. Hence, $\hat{\alpha}$ satisfies (FPCfix), and is therefore in $\mathbb{A}$. We show that $\mathcal{L}(\hat{\alpha}, \nu) > \mathcal{L}(\alpha, \nu)$, contradicting (C):

$$\begin{aligned}\int_S \big[\mathcal{W}(s, \hat{w}) - \nu\big] d\alpha(s) &= \int_{s_*(\nu)}^{\tilde{s}} \big[\mathcal{W}(s, \hat{w}) - \nu\big] d\alpha(s) + \int_{\tilde{s}}^{s^*(\nu)} \big[\mathcal{W}(s, \hat{w}) - \nu\big] d\alpha(s) \\ &\leq \big[\mathcal{W}(\tilde{s}, \hat{w}) - \nu\big] \int_{s_*(\nu)^+}^{\tilde{s}} d\alpha(s) + \int_{\tilde{s}}^{s^*(\nu)} \big[\mathcal{W}(s, \hat{w}) - \nu\big] d\alpha(s) \\ &= \big[\mathcal{W}(\tilde{s}, \hat{w}) - \nu\big] \Big\{\alpha(\tilde{s}) - \alpha\big(s_*(\nu)^+\big)\Big\} + \int_{\tilde{s}}^{s^*(\nu)} \big[\mathcal{W}(s, \hat{w}) - \nu\big] d\alpha(s),\end{aligned}$$

where the first inequality uses the equality $\mathcal{W}\big(s_*(\nu)\big) = \nu$ to eliminate the atom at $s_*(\nu)$, and the fact that $\mathcal{W}(\cdot, \nu)$ is increasing to the left of $\hat{s}$ by Lemma 1.

By definition of $\hat{\alpha}$, the first term is strictly dominated by

$$\begin{aligned}\big[\mathcal{W}(\tilde{s}, \hat{w}) - \nu\big] \Big\{\alpha(\tilde{s}) - \alpha\big(s_*(\nu)^+\big) + \eta\Big\} &= \big[\mathcal{W}(\tilde{s}, \hat{w}) - \nu\big] \Big\{\hat{\alpha}(\tilde{s}) - \hat{\alpha}\big(s_*(\nu)^+\big)\Big\} \\ &= \int_{s_*(\nu)}^{\tilde{s}} \big[\mathcal{W}(s, \hat{w}) - \nu\big] d\hat{\alpha}(s),\end{aligned}$$

while the second term is equal to

$$\int_{\tilde{s}}^{s^*(\nu)} \big[\mathcal{W}(s, \hat{w}) - \nu\big] d\hat{\alpha}(s).$$

Hence, $\int_S \big[\mathcal{W}(s, \hat{w}) - \nu\big] d\alpha(s) < \int_S \big[\mathcal{W}(s, \hat{w}) - \nu\big] d\hat{\alpha}(s)$, which contradicts (C). To rule out discontinuities at $s^*(\nu)$, we follow a similar construction. Therefore, (C) implies (ii). $\square$

Optimal allocation rule under (UID). As in the standard case, we consider the following relaxation of the simplified problem:

$$\max_{\alpha} \int_{s_*}^{\hat{s}} \alpha(s)\{w(s) - \hat{w}\}dF(s) + \int_{\hat{s}}^{s^*} \alpha(t)\{w(t) - \hat{w}\}dF(t) \qquad (\overline{\text{Relax BP}})$$
$$\text{s.t. } \alpha(t) - \alpha(s) \le \bar{c}(t|s), \quad \forall s_* \le s \le \hat{s} \le t \le s^*, s < t$$

in which we only require falsification-proofness to prevent ineligible scores from falsifying to eligible targets, and separate the objective function between ineligible and eligible scores.

We perform the same transformations on this program as in the standard case with one small difference. We change the variables of this problem to identify scores by their distance to the eligibility threshold, letting $y = \hat{s} - s$ for $s \le \hat{s}$, and $z = t - \hat{s}$ for $t \ge \hat{s}$. These variables belong, respectively, to the space $Y = [0, \hat{s} - s_*]$, and to the space $Z = [0, s^* - \hat{s}]$. By (ZAS), each of these spaces harbors the same total mass of surplus. We endow each of them with a probability distribution measuring the fraction of this total mass of surplus, as given by the cumulative density functions

$$P(y) = \frac{\mathcal{W}(\hat{s}, \hat{w}) - \mathcal{W}(\hat{s} - y, \hat{w})}{\mathcal{W}(\hat{s}, \hat{w}) - \mathcal{W}(s_*, \hat{w})}, \qquad \text{and} \qquad Q(z) = \frac{\mathcal{W}(\hat{s}, \hat{w}) - \mathcal{W}(\hat{s} + z, \hat{w})}{\mathcal{W}(\hat{s}, \hat{w}) - \mathcal{W}(s^*, \hat{w})},$$

where the normalizing factor is the total mass. Note that $dP(y) \propto |w(\hat{s} - y) - \hat{w}|dF(\hat{s} - y)$, and $dQ(z) \propto |w(\hat{s} + z) - \hat{w}|dF(\hat{s} + z)$.

Finally, the only difference comes in the definition of prices. We rewrite the allocation probabilities as location-specific prices, $\phi(y) = \alpha(\hat{s} - y) + k/2$ and $\psi(z) = \alpha(\hat{s} + z) - k/2$, so the program becomes (up to multiplication by the normalizing factor)

$$\max_{\phi,\psi} \int_Z (\psi(z) + k/2)dQ(z) - \int_Y (\phi(y) - k/2)dP(y)$$
$$\text{s.t.} \quad \psi(z) - \phi(y) \le c(\hat{s} + z|\hat{s} - y) \quad \forall y, z.$$

Since

$$\int_Z (\psi(z) + k/2)dQ(z) - \int_Y (\phi(y) - k/2)dP(y) = \int_Z \psi(z)dQ(z) - \int_Y \phi(y)dP(y) + k,$$

solving this program is equivalent to solving the program:

$$\begin{aligned} \max_{\phi,\psi} \quad & \int_Z \psi(z)dQ(z) - \int_Y \phi(y)dP(y) \\ \text{s.t.} \quad & \psi(z) - \phi(y) \le c(\hat{s}+z|\hat{s}-y) \quad \forall y,z, \end{aligned}$$

which is exactly the same optimal transport problem we solve in the standard case. Hence, the formulas for the optimal prices ϕ and ψ are identical, and only the mapping from prices ϕ, ψ to allocation probability changes.

Therefore, we obtain the same formula for the optimal solution to (P). However, there is an important difference, which is the introduction of a jump of size k at the eligibility threshold $\hat{s}$. The jump is generated by the boundary condition below

$$\bar{\alpha}^*_{uid}(s,\hat{w},r) = \begin{cases} 0 & \text{if } s < s_* \\ \bar{\Gamma}_{uid} I(\hat{w},r) - \int_{s_*}^{s} c_s\big(m(x)|x\big)dx & \text{if } s \in [s_*, \hat{s}) \\ 1 - \bar{\Gamma}_{uid}\bar{I}(\hat{w},r) - \int_{s}^{s^*} c_t\big(x|m^{-1}(x)\big)dx & \text{if } s \in [\hat{s}, s^*] \\ 1 & \text{if } s > s^* \end{cases}.$$

The value of $\alpha^*(\hat{s},\hat{w},r)$ is irrelevant, and we have chosen it to make $\bar{\alpha}^*_{uid}$ right-continuous.

The growth interval $[s_*, s^*]$ is uniquely determined by the *boundary condition*

$$s_* = \min\big\{s \in [s_*(0), \hat{s}] : \bar{c}\big(m(s)|s\big) \le 1\big\}. \qquad (\overline{\text{B}})$$

We are using $\bar{c}$ instead of c for the new boundary condition, which generates the jump at $\hat{s}$.

Whether the probability constraint binds or not depends on the magnitude of falsification costs, and its degree of slackness is measured by the *probability gap*, now defined as

$$\bar{\Gamma}_{uid} = 1 - \bar{c}(s^*|s_*). \qquad (\overline{\text{Gap}})$$

Then, the probability constraint is slack, and $\bar{\Gamma}_{uid} > 0$, if and only if the *slackness condition* $\bar{c}\big(s^*(0)|s_*(0)\big) < 1$ is satisfied.

Given the boundary condition ($\overline{\text{B}}$), the probability constraint is slack, and the gap positive $\bar{\Gamma}_{uid} > 0$ in the neutral probability case, if and only if the *slackness condition* $\bar{c}\big(\overline{s}, \underline{s}\big) < 1$ is satisfied. When it holds, there is a gap $\bar{\Gamma}_{uid} > 0$ between the total growth of the optimal rule and its upper bound (equal to 1). Multiplicity, therefore, arises under the condition:

$$\hat{w} = \bar{w} \text{ and } \bar{c}(\bar{s}|\underline{s}) < 1. \tag{$\overline{\text{Mult}}$}$$

Theorem S1.1 (Optimal rule under UID with a fixed cost)**.** *If* ($\overline{\text{Mult}}$) *does not hold, then* $\bar{\alpha}^*_{uid}$ *is independent of* r*, and it is the unique solution of* (P)*. If* ($\overline{\text{Mult}}$) *holds, then the set of optimal rules is* $\left\{\bar{\alpha}^*_{uid}(\cdot, \bar{w}, r)\right\}_{r\in[0,1]}$.

Before proving the theorem, some comments are in order. Note that $\bar{\alpha}^*_{uid}$ closely resembles α^*_{uid}. The important difference is in the statement of the boundary condition ($\overline{\text{B}}$) and the definition of the probability gap ($\overline{\text{Gap}}$). The boundary condition changes the nature of the growth interval, generating a jump between the eligible and ineligible sides of the optimal rule, whereas the growth of the allocation rule on each of these sides is exactly as it is without a fixed cost. It is clear from the structure that increasing the fixed cost k reduces the size of the growth interval and increases both allocative surplus and the welfare of the agents. Naturally, when $k \geq 1$, the optimal falsification rule is the first-best allocation rule, which becomes falsification-proof because of the high fixed cost.

The proof of the theorem closely follows the proof of Theorem 2 in the main paper.

Proof of Theorem S1.1. To keep notation simple, we only indicate the dependence of $\bar{\alpha}^*_{uid}$ on $\hat{w}, r$ when it is useful for the argument. We check that the conditions of Lemma S1.1 are satisfied. We pick the multiplier $\nu = \mathcal{W}(s_*, \hat{w}) \geq 0$. Then $\bar{\alpha}^*_{uid}$ clearly satisfies (i) and it satisfies (ii) by construction. To see that it satisfies (iii) and (iv), note that $\bar{\alpha}^*_{uid}(s^*) - \bar{\alpha}^*_{uid}(s_*) = \bar{c}(s^*|s_*)$ since (FPCfix) is binding for (s_*, s^*). Hence, by ($\overline{\text{B}}$), either $\bar{\alpha}^*_{uid}(s^*) - \bar{\alpha}^*_{uid}(s_*)$ is equal to 1 and the probability constraint is binding, or it is strictly less than 1, and then $\nu = 0$ and $(s_*, s^*) = (s_*(0), s^*(0))$.

By the optimal transport connection established above, $\bar{\alpha}^*_{uid}$ solves the relaxed program ($\overline{\text{Relax BP}}$). To show that it satisfies (v), we need to show that it satisfies (FPCfix) in (ii) for any pair s, t such that $s, t \in [s_*, \hat{s}]$ or $s, t \in [\hat{s}, s^*]$. Take, for example, the first case. Then

$$\begin{aligned}
\bar{\alpha}^*_{uid}(t) - \bar{\alpha}^*_{uid}(s) &= -\int_s^t c_s\big(m(x)|x\big)dx \leq -\int_s^t c_s(m(t)|x)dx && \text{(by UID)} \\
&= c\big(m(t)|s\big) - c\big(m(t)|t\big) \leq c(t|s) - c(t|t) = c(t|s). && \text{(by UID)}
\end{aligned}$$

The argument is similar in the second case.

For uniqueness, first note that $\bar{c}(s^*(\nu)|s_*(\nu))$ is increasing in ν so there is a single value of ν that satisfies ($\overline{\text{B}}$), and hence satisfies the necessary and sufficient conditions of Lemma S1.1. Then for this ν and the corresponding bounds (s_*, s^*), the solution to the optimal transport problem is uniquely determined up to a constant. This constant is uniquely determined either by the probability constraint if it binds, that is, if $\bar{c}(s^*|s_*) = 1$, or by the requirement that $\bar{\alpha}^*_{uid}(s_*) = 0$ under low-score priority, and $\bar{\alpha}^*_{uid}(s^*) = 1$ under high-score priority. Uniqueness fails only if we are in the neutral priority case where $\bar{w} = \hat{w}$ and the probability constraint is slack. In this case, $(s_*, s^*) = (s_*(0), s^*(0)) = (\underline{s}, \bar{s})$. Hence, for the probability constraint not to bind, it must be the case that $\bar{c}(\bar{s}|\underline{s}) < 1$. The designer is then indifferent across all allocation rules $\bar{\alpha}^*_{uid}(s, \bar{w}, r)$ for any $r \in [0, 1]$. Indeed, for $r' > r$, we have $\bar{\alpha}^*_{uid}(s, \bar{w}, r') - \bar{\alpha}^*_{uid}(s, \bar{w}, r) = (r' - r)\bar{\Gamma}_{uid}$, so the difference in the designer's payoff is $(r' - r)\bar{\Gamma}_{uid} \int_{\underline{s}}^{\bar{s}} \{w(s) - \hat{w}\} dF(s) = (r' - r)\bar{\Gamma}_{uid}(\bar{w} - \hat{w}) = 0$. □

S2 Heterogeneity and multiple groups

In the paper, we assume that the only source of heterogeneity across agents is their score. However, the model extends to a richer heterogeneity structure with both observable and unobservable dimensions. On the one hand, unobservable dimensions cannot be elicited by any mechanism, and the falsification-proofness constraint requires preventing falsification between any two scores under the worst-case scenario (i.e., for the type of agents for whom it is the least costly). Therefore, the problem with additional unobservable heterogeneity is similar to the one with no additional heterogeneity, but where agents have the worst-case falsification cost function. On the other hand, observable heterogeneity leads to the creation of groups of agents within which the optimal allocation problem is identical to the setup of the paper.

S2.1 Model

Framework. Each agent is characterized by a private type $\theta = (i, s, k)$ and their *worth* w. Agents may know their worth (if θ is a sufficient statistic for w), or not, with no effect on the results.

The first dimension of the type $i \in I$ encompasses all relevant publicly observable and unfalsifiable characteristics of an agent. We refer to i as an agent's *group*, and assume I is a finite set. The mass of group i is $\mu_i > 0$, where $\sum_{i \in I} \mu_i = 1$. The second dimension of an agent's type is the natural score $s \in S_i \subseteq \mathbb{R}$. The last dimension of type, $k \in K_{i,s}$, is a vector of privately known characteristics that includes an agent's value for the good $v(k) > 0$, information about their individual falsification cost, and possibly other characteristics correlated with their worth.

Distributional assumptions. Each agent draws a vector of characteristics (θ, w) i.i.d. from a joint distribution. Hence, the different dimensions of an agent's vector of characteristics can be, and typically are, correlated; but they are independent of other agents' characteristics. F_i denotes the cumulative distribution function of natural score conditional on i, which we assume to have full support on an interval $S_i = [\underline{s}_i, \overline{s}_i]$, and no atoms. Conditional on (i, s), the remainder of the type vector is fully supported on $K_{i,s}$.

Designer and agent payoffs. We assume that the worth w is bounded and integrable conditional on (i, s). We denote the corresponding expected worth by $w_i(s) = \mathbb{E}(w|i, s)$, and by $\bar{w}_i = \mathbb{E}(w|i)$ the expected worth in group i. The designer's payoff from assigning an object to a group i agent with score s is $w_i(s)$. We assume that score and worth are positively related in the sense that, for every group i, $w_i(s)$ is strictly increasing. The expected payoff of an agent is $\alpha v - C(t, \theta)$, where α is the probability of getting an object, and $C(t, \theta) \geq 0$ defines the cost for type θ to produce a score t.

Not falsifying is costless, so $C(s, \theta) = 0$ for an agent of type $\theta = (i, s, k)$. The cost of producing score t depends not only on the natural score s, but also on k. The falsification cost may reflect technical costs, psychological lying costs, and expected penalties.

Falsification-proof mechanisms. We restrict the designer to *falsification-proof mechanisms*, that is, mechanisms that incentivize agents to produce their natural scores. Under this assumption, we show in Perez-Richet and Skreta (2023) that it is without loss of generality to restrict attention to *score-based allocation rules* $\boldsymbol{\alpha} = (\alpha_i)_{i \in I}$, where $\alpha_i : S_i \to [0, 1]$ is the probability that an object is allocated to an agent from group i conditional on their produced score. Such rules only condition on the observable dimension i, and the score s, but not on k. The reason is that once we restrict attention to mechanisms that recommend agents to produce their natural score (which by definition involves deterministic recommendations), agents with the same score s will pool on reporting the (k, v) pair that maximizes their allocation probability, so the designer cannot elicit this information.[1] Among agents from the

[1]Lemma 1 in Akbarpour, Dworczak and Kominers (2024) makes a similar point in a setting where all dimensions are costless to misrepresent, and transfers are allowed: the principal can only elicit

same group and with the same natural score, falsification is most tempting for those with the highest valuation and lowest falsification costs. Therefore, a mechanism is falsification-proof if and only if it satisfies the following constraint

$$\forall(i,s,t), \quad \alpha_i(t) - \alpha_i(s) \le c_i(t|s), \tag{FPC}$$

where $c_i(t|s) = \inf_{k\in K_{is}} \frac{1}{v(k)} C(t,i,s,k)$ is the *least cost* for a member of group i with natural score s to falsify to t.[2] We refer to c_i as the *least-cost* or simply cost function for group i.

Falsification cost. We assume that each group's least-cost function satisfies the assumptions we make on the single cost function in the paper. In particular, we assume that each least-cost function satisfies either (UID) or (UDD).

Allocative constraints. In many applications we consider, allocative constraints are irrelevant because prizes are immaterial, such as services, certificates, labels or awards. To cover other applications, we allow for a *resource constraint* and *quota constraints*. The resource constraint requires

$$\sum_i \mu_i \int_{\underline{s}_i}^{\overline{s}_i} \alpha_i(s) dF_i(s) \le \bar{\rho}. \tag{RC}$$

In addition, the designer may have to satisfy a system of exogenous quotas $\phi = (\phi_i)_{i\in I}$, where $\phi_i \in [0,1]$ is a fraction of objects reserved for group i, with $\sum_i \phi_i \le 1$, and $\phi_i\bar{\rho} \le \mu_i$. The quota constraints are

$$\forall i, \quad \mu_i \int_{\underline{s}_i}^{\overline{s}_i} \alpha_i(s) dF_i(s) \ge \phi_i\bar{\rho}. \tag{QC}$$

A mechanism is feasible if it satisfies these *allocative constraints*. If $\bar{\rho} = 1$ and $\phi = 0$, the problem has no allocative constraints. Feasible mechanisms also need to satisfy probability constraints:

$$\forall(i,s) \quad 0 \le \alpha_i(s) \le 1. \tag{PC}$$

information on the dimensions that correlate with agents' willingness to pay.

[2]This infimum exists because $C(t,\theta)$ is bounded below by 0. We assume that this bound is tight in the sense that, for every i,s,t and every $\varepsilon > 0$, there exists a strictly positive mass of agents from group i with natural score s whose cost of falsifying to t is lower than $c_i(t|s) + \varepsilon$.

Designer's program. The restriction to falsification-proof mechanisms implies the agent's observed score is the natural one, so α_i writes as function on the natural score s rather than falsified scores. The designer's program is to choose a score-based allocation rule $\boldsymbol{\alpha}$ that solves:

$$\max_{(\alpha_i)_{i\in I}} \sum_i \mu_i \int_{S_i} w_i(s)\alpha_i(s)dF_i(s) \quad \text{s.t.} \quad (\text{PC}), (\text{FPC}), (\text{RC}), (\text{QC}). \tag{P}$$

To solve the designer's problem, we decompose it into a collection of group specific *within problems*, that consist in optimally allocating a fixed mass of objects within a group, and an overall *across problem* of optimally choosing the masses of objects accruing to the different groups while satisfying the allocative constraints.

Within problem. Let ρ_i be the mass of objects allocated to group i. Then the corresponding within problem is

$$W_i(\rho_i) = \max_{\alpha_i} \int_{S_i} \alpha_i(s)w_i(s)dF_i(s) \tag{P$_W$}$$

$$\text{s.t. } (\text{FPC}), (\text{PC}),$$

$$\mu_i \int_{S_i} \alpha_i(s)dF_i(s) = \rho_i, \tag{RC$_W$}$$

where the within resource constraint (RC$_W$) must hold with equality. Since we require the whole mass ρ_i to be allocated, the within problem is feasible only if $\rho_i \leq \mu_i$, hence its value function is equal to $-\infty$ otherwise.

Across problem. A *group allocation profile* $\boldsymbol{\rho} = (\rho_i)_{i\in I}$ satisfies the allocative constraints if it belongs to the feasible set $R = \{\boldsymbol{\rho} : \sum_i \rho_i \leq \bar{\rho},\ \rho_i \geq \phi_i\bar{\rho}\ (\forall i)\}$. The designer's problem is then summarized by the across problem

$$\overline{W}(\boldsymbol{F}, \boldsymbol{c}) = \max_{\boldsymbol{\rho}\in R} \sum_i \mu_i W_i(\rho_i), \tag{P$_A$}$$

where $\boldsymbol{F} = (F_i)_{i\in I}$ and $\boldsymbol{c} = (c_i)_{i\in I}$ denote profiles.

S2.2 Optimal within group allocation

We first derive an optimal allocation rule for the within problem. To simplify notation, we drop the group index i. Let $\hat{w}/\mu$ be the Lagrange multiplier on the resource constraint, where μ is the size of the group. The Lagrangian for (P_W) is then

$$\int_S \alpha(s)\{w(s) - \hat{w}\}dF(s) + \hat{w}\frac{\rho}{\mu}.$$

Maximizing the Lagrangian for a fixed value of the multiplier $\hat{w}/\mu$ is therefore equivalent to solving our baseline problem in the paper (P) with outside option $\hat{w}$. To solve the within problem, we then need to identify the value of the Lagrange multiplier, i.e., of the outside option, that makes (RC_W) hold. To do this, we first study how the baseline allocation $\alpha^*(\cdot, \hat{w}, r)$ varies with the outside option $\hat{w}$, and the neutral gap share r.

The mechanics of indirect effects. Since adjusting outside options and neutral gap shares are the key tools to achieve particular group allocations, understanding how the baseline allocation reacts to such changes reveals the mechanics of the indirect effects.

We show in Appendix D, Proposition D.1 that the total mass of objects allocated under $\alpha^*(\cdot, \hat{w}, r)$ is smoothly decreasing in the outside option $\hat{w}$ and increasing in the neutral gap share r when (Mult) holds, so we can use these two levers to smoothly shift the mass of allocated objects.

Intuitively, a higher value of the outside option should reduce the mass of allocated objects in the baseline problem. In fact, a stronger result holds, since the effect is uniform across all scores: for every s, the baseline allocation probability is decreasing in $\hat{w}$. A higher value of the neutral gap share naturally increases the baseline allocation probability for all scores when it is effective, that is, under (Mult).

Let $A^*(\hat{w}, r) = \int_S \alpha^*(s, \hat{w}, r)dF(s)$ denote the per capita mass of objects allocated to the group under the baseline rule $\alpha^*(s, \hat{w}, r)$. Proposition D.1 implies Corollary D.1 which states that $A^*(\hat{w}, r)$ is strictly decreasing in $\hat{w}$, continuous in $\hat{w}$ and independent of r unless (Mult) holds, in which case $A^*(\bar{w}, r)$ is strictly increasing and continuous in r.

Optimal within allocation. Returning to the within problem, the outside option $\hat{w}$ is equal to the Lagrange multiplier on the resource constraint, scaled by group size μ, and can be interpreted as the shadow price of marginally tightening the constraint. To ensure that (RC_W) holds, we must adjust $\hat{w}$ and r so that $A^*(\hat{w}, r) = \rho/\mu$. Theorem S2.1 shows that finding such values of $\hat{w}$ and r is always possible:

Theorem S2.1 (Optimal within group allocation)**.** *For any $0 \leq \rho \leq \mu$, there exists a unique outside option value $\hat{w}(\rho)$ and, under* (Mult)*, a unique neutral gap share $r(\rho)$, such that $\mu A^*\big(\hat{w}(\rho), r(\rho)\big) = \rho$. Furthermore, $\hat{w}(\rho)$ is continuous, decreasing in ρ unless* (Mult) *holds, in which case it is constant at $\bar{w}$. The function $r(\rho)$ is continuous and strictly increasing. The baseline allocation rule $\alpha^*(s, \hat{w}(\rho), r(\rho))$ is then the unique solution to the within problem* (P_W)*. The value function of* (P_W)*, $W(\rho)$ is strictly concave in ρ unless* (Mult) *holds.*

To see why Theorem S2.1 holds, note that, by assumption, $w(s)$ is bounded. It is easy to see that, for any $\hat{w}$ below the lower bound on $w(s)$, the unique baseline allocation rule allocates with certainty to all scores, regardless of scores. Similarly, for any $\hat{w}$ above the upper bound on $w(s)$, the baseline allocation rule never allocates any object. Hence, by varying $\hat{w}$ between these bounds, we can find an outside option $\hat{w}(\rho)$ such that the baseline allocation rule satisfies the resource constraint, and therefore solves (P_W). If $\hat{w}(\rho) = \bar{w}$ and $c(\bar{s}|\underline{s}) < 1$, we also need to adjust r to a unique value $r(\rho)$ so as to allocate exactly ρ objects. The allocation rule $\alpha^*(s, \hat{w}(\rho), r(\rho))$ is then the unique solution to the within problem.

S2.3 Optimal across-group allocation

We characterize the solution to the across problem and provide an algorithm to determine the optimal allocation profile $\boldsymbol{\rho} = (\rho_i)_{i \in I}$.

Theorem S2.2 (Optimal across group allocation)**.** *The across problem* (P_A) *admits a solution $\boldsymbol{\rho}$. Furthermore, $\boldsymbol{\rho}$ solves the across problem if and only if there exist a scalar $\lambda_R \geq 0$ and, for each i, a scalar $\lambda_i \geq 0$, an outside option value $\hat{w}_i(\rho_i)$, and a neutral gap share $r_i(\rho_i)$ such that:*

(i) $\lambda_i(\phi_i \bar{\rho} - \rho_i) = 0$ *for all i*,

(ii) $\lambda_R\big(\sum_i \rho_i - \bar{\rho}\big) = 0$,

(iii) $\hat{w}_i(\rho_i) = \lambda_R - \lambda_i$,

(iv) $\mu_i A_i^*\big(\hat{w}_i(\rho_i), r_i(\rho_i)\big) = \rho_i$.

The solution $\boldsymbol{\rho}$ is unique if, for each i, (Mult) *does not hold at $\hat{w}_i(\rho_i)$.*

The characterization in Theorem S2.2 suggests the following algorithm to find a solution to the across problem. First, we compute the solutions to each within problem, setting all outside options to 0. For these solutions, we check which constraints are binding or violated. Next, we adjust outside options to satisfy all the previously violated constraints with equality when recomputing the corresponding solution. This may lead to hitting additional constraints. Indeed, increasing allocation to one group to satisfy its quota may lead to the violation of a previously slack resource constraint, or another group's quota constraint if the resource constraint was already binding. If so, we adjust outside options to satisfy with equality all constraints that were binding or violated at any previous step. Since, at any step, the set of constraints that have required adjustment at some previous step is bounded, and necessarily increases with each step, the process must eventually end. The allocation profile at which it ends is a solution to the across problem. In the next subsection, we state the algorithm formally and show that it finds a solution to the across problem.

S2.4 Algorithm for the across problem

We present an algorithm that finds a solution to the across problem. We extend the definition of $\hat{w}_i^\phi$ as the unique value of $\hat{w}$ such that $\mu_i A_i^*(\hat{w}, r) = \phi_i \bar{\rho}$ for some r, and let r_i^ϕ be the unique value of r that satisfies this equality if $\hat{w}_i^\phi = \bar{w}_i$ (otherwise let r_i^ϕ be any value on $[0, 1]$).

Algorithm 1: Algorithm to solve the across problem

```
∀i,  ρ_i^0 ← μ_i A_i^*(0, 1);
R^0 ← 𝓡(ρ^0);
Q^0 ← 𝓠(ρ^0);
k ← 0;
repeat
    k ← k + 1;
    ∀ℓ ∈ Q^{k−1},  ŵ_ℓ^k ← ŵ_ℓ^φ and r_ℓ^k ← r_ℓ^φ;
    if R^{k−1} = 0 then
        ∀ℓ ∉ Q^{k−1},  ŵ_ℓ^k ← 0 and r_ℓ^k ← 1;
    else
        ŵ, r ← Solution of: Σ_{ℓ∈Q^{k−1}} φ_ℓ ρ̄ + Σ_{i∉Q^{k−1}} μ_i A_i^*(ŵ, r 𝟙_{ŵ=w̄_i}) = ρ̄;
        ∀ℓ ∉ Q^{k−1},  ŵ_ℓ^k ← ŵ and r_ℓ^k ← r 𝟙_{ŵ=w̄_ℓ};
    end
    ∀i,  ρ_i^k ← μ_i A_i^*(ŵ_i^k, r_i^k);
    R^k ← 𝓡(ρ^k);
    Q^k ← 𝓠(ρ^k);
until Q^k = Q^{k−1} and R^k = R^{k−1};
```

We did not specify how to find a solution $(\hat{w}, r)$ to

$$\sum_{\ell \in Q^{k-1}} \phi_\ell \bar{\rho} + \sum_{i \notin Q^{k-1}} \mu_i A_i^*(\hat{w}, r\mathbb{1}_{\hat{w}=\bar{w}_i}) = \bar{\rho}$$

within the algorithm. Note, however, that the left-hand side of the equation can be decreased continuously by continuously raising $\hat{w}$ whenever $\hat{w} \neq \bar{w}_i$, for all i, and by continuously decreasing r from 1 to 0 and keeping $\hat{w}$ constant whenever $\hat{w} = \bar{w}_i$ for some i. Therefore a simple algorithm can solve this equation.

Proposition S2.1. *Algorithm 1 finds a solution of* (P_A) *in finitely many steps.*

Proof. The sequence (Q^k, R^k) is increasing and bounded above by $(I, 1)$ in the $(\subseteq, \leq)$ order on $2^I \times \{0, 1\}$, so the algorithm stops in finitely many steps. Let k be the step at which it stops. Let $\lambda_R = \hat{w}_i^k$ and $\lambda_i = 0$ for all $i \notin Q^k$. This is consistent since $\hat{w}_i^k$ must be equal across all $i \notin Q^k$. Let $\lambda_\ell = \lambda_R - \hat{w}_\ell^k$ for $\ell \in Q^k$. Then it is easy to verify the vector of multipliers $\boldsymbol{\lambda}$, $\boldsymbol{\rho}^k$, $\hat{w}^k$ and $\boldsymbol{r}^k$ satisfy all the conditions of Theorem S2.2. Therefore $\boldsymbol{\rho}^k$ is a solution to the across problem. □

S2.5 Comparative statics and welfare results

Designer welfare. We show that designer welfare is decreasing in gaming ability, and increasing with first-order stochastic dominance shifts of the score distribution. Let $\boldsymbol{c}^\gamma$ denote a profile of parameterized cost functions $c_i^{\gamma_i} = \frac{1}{\gamma_i} c_i(t|s)$.

Proposition S2.2 (Properties of the designer's value function)**.** *The value function of the across problem, $\overline{W}(\boldsymbol{F}, \boldsymbol{c}^\gamma)$, is nonincreasing in γ_i, and nondecreasing in F_i with respect to the first-order stochastic dominance order.*

Proof of Proposition S2.2. Increasing γ_i lowers falsification costs, which tightens (FPC) and therefore shrinks the set of feasible allocation rules in the original problem, therefore weakly decreases its value function $\overline{W}$. Suppose $\tilde{F}_i$ first-order stochastically dominates F_i, and let $F_i^x = x\tilde{F}_i + (1-x)F_i$. Then F_i^x increases with x in the FOSD order.

Consider the within problem for group i under the score distribution F_i^x. To clarify the dependence on x, we denote its value function by $W_i(\rho_i|x)$ in this proof. By Lemma D.1,

$$W_i(\rho_i|x) = \min_{\hat{w} \in [w^-, w^+]} \max_{\alpha} \int_{S_i} \alpha(s)\{w_i(s) - \hat{w}\} dF_i^x(s) + \hat{w}\rho_i/\mu_i,$$

and $\left(\alpha_i^*(\cdot,\hat{w}_i(\rho_i),r_i(\rho_i)|x),\hat{w}_i(\rho_i)\right)$ is the unique solution to this saddle-point problem. In what follows, let $\alpha_i^*(s)$ denote the function $\alpha_i^*(\cdot,\hat{w}_i(\rho_i),r_i(\rho_i)|x)$.

Let $\mathcal{L}(\alpha,\hat{w},x)=\int_{S_i}\alpha(s)\{w_i(s)-\hat{w}\}dF_i^x(s)+\hat{w}\rho_i/\mu_i$ be the objective function. It is continuously differentiable in x since it is linear. Furthermore, the saddle-point problem admits a solution for every $x\in[0,1]$ by Theorem S2.1. The interval $[w^-,w^+]$ and the space of nondecreasing continuous functions in which α is taken is also compact by Helly's selection theorem. Therefore, we can apply the envelope theorem for saddle-points of Milgrom and Segal (2002, Theorem 5), and our uniqueness result to obtain that $W_i(\rho_i|x)$ is differentiable in x, and

$$\begin{aligned}\frac{\partial W_i(\rho_i|x)}{\partial x}&=\frac{\partial\mathcal{L}\Big(\alpha_i^*(\cdot,\hat{w}_i(\rho_i),r_i(\rho_i)),\hat{w}_i(\rho_i),x\Big)}{\partial x}\\&=\int_{S_i}\alpha_i^*(s)\{w_i(s)-\hat{w}_i(\rho_i)\}d\tilde{F}_i(s)-\int_{S_i}\alpha_i^*(s)\{w_i(s)-\hat{w}_i(\rho_i)\}dF_i(s).\end{aligned}$$

Using the differential form, we have:

$$\frac{\partial W_i(\rho_i|x)}{\partial x}=\int_{S_i}\alpha_i^{*'}(s)\left(\tilde{\mathcal{W}}_i(s,\hat{w}_i(\rho_i))-\mathcal{W}_i(s,\hat{w}_i(\rho_i))\right)ds.$$

Next, we show this must be nonnegative. Indeed, note first

$$\begin{aligned}\tilde{\mathcal{W}}_i^+(s,\hat{w}_i(\rho_i))-\mathcal{W}_i^+(s,\hat{w}_i(\rho_i))=&\big(1-\tilde{F}_i(s)\big)\int_s^{\bar{s}_i}\big\{w_i(y)-\hat{w}_i(\rho_i)\big\}d\frac{\tilde{F}_i(y)}{1-\tilde{F}_i(s)}\\&-\big(1-F_i(s)\big)\int_s^{\bar{s}_i}\big\{w_i(y)-\hat{w}_i(\rho_i)\big\}d\frac{F_i(y)}{1-F_i(s)}.\end{aligned}$$

By first-order stochastic dominance, $1-\tilde{F}_i(s)\geq 1-F_i(s)$, and the stochastic dominance ordering of the conditional distributions on $[s,\bar{s}_i]$ is preserved since $\frac{\tilde{F}_i(y)}{1-\tilde{F}_i(s)}\leq\frac{F_i(y)}{1-F_i(s)}$. Since $w_i(\cdot)$ is an increasing function, this implies

$$\tilde{\mathcal{W}}_i^+(s,\hat{w}_i(\rho_i))-\mathcal{W}_i^+(s,\hat{w}_i(\rho_i))\geq 0.$$

If $\bar{w}_{F_i}\leq\bar{w}_{\tilde{F}_i}\leq\hat{w}_i(\rho_i)$, then the group has low-score or neutral priority under both distributions. Using the differential version of the objective function (DOF), the difference in welfare is given by

$$\int_{\underline{s}_i}^{\bar{s}_i}\alpha_i^{*'}(s)\big\{\tilde{\mathcal{W}}_i^+(s,\hat{w}_i(\rho_i))-\mathcal{W}_i^+(s,\hat{w}_i(\rho_i))\big\}ds.$$

Since $\alpha_i^{*'}(s) \geq 0$, and the difference in cumulative surplus is positive, then $\frac{\partial W_i(\rho_i|x)}{\partial x} \geq 0$.

Suppose instead, $\bar{w}_{F_i} \leq \hat{w}_i(\rho_i) \leq \bar{w}_{\tilde{F}_i}$, so the shift in score distributions switches the priority of the group. Then, using (DOF) for F_i, ($\overline{\text{DOF}}$) for $\tilde{F}_i$, and the relationship $\mathcal{W}^-(s,\hat{w}) = \mathcal{W}^+(s,\hat{w}) - (\bar{w} - \hat{w})$, we can write the welfare change as

$$\left(\bar{w}_{\tilde{F}_i} - \hat{w}_i(\rho_i)\right)\left(1 - \int_{\underline{s}_i}^{\bar{s}_i} \alpha_i^{*'}(s)ds\right)\int_{\underline{s}_i}^{\bar{s}_i} \alpha_i^{*'}(s)\left\{\tilde{\mathcal{W}}_i^+(s,\hat{w}_i(\rho_i)) - \mathcal{W}_i^+(s,\hat{w}_i(\rho_i))\right\}ds,$$

where the second term is positive for the same reasons as in the previous case, and the first term is equal to $\left(\bar{w}_{\tilde{F}_i} - \hat{w}_i(\rho_i)\right)\alpha_i^{*'}(\underline{s}_i) \geq 0$. Hence, again, $\frac{\partial W_i(\rho_i|x)}{\partial x} \geq 0$.

Suppose finally $\hat{w}_i(\rho_i) \leq \bar{w}_{F_i} \leq \bar{w}_{\tilde{F}_i}$ so the priority is to high scores under both distributions. Then, using ($\overline{\text{DOF}}$) and the relationship $\mathcal{W}^-(s,\hat{w}) = \mathcal{W}^+(s,\hat{w}) - (\bar{w} - \hat{w})$, we can write the welfare change as

$$\left(\bar{w}_{\tilde{F}_i} - \bar{w}_{F_i}\right)\left(1 - \int_{\underline{s}_i}^{\bar{s}_i} \alpha_i^{*'}(s)ds\right)\int_{\underline{s}_i}^{\bar{s}_i} \alpha_i^{*'}(s)\left\{\tilde{\mathcal{W}}_i^+(s,\hat{w}_i(\rho_i)) - \mathcal{W}_i^+(s,\hat{w}_i(\rho_i))\right\}ds,$$

and both terms are positive, and then $\frac{\partial W_i(\rho_i|x)}{\partial x} \geq 0$.

Now, consider the across problem. Applying the (classical) envelope theorem to this problem, and letting $\boldsymbol{\rho}^*$ denote its unique solution, we obtain

$$\frac{\partial \overline{W}(x)}{\partial x} = \mu_i \frac{\partial W_i(\rho_i^*|x)}{\partial x} \geq 0.$$

□

Any increase in gaming ability tightens (FPC), thereby reducing welfare. The effect of score distributions is more difficult to analyze, as a first-order stochastic dominance shift in the score distribution of one group may have a non-obvious direct effect on the baseline allocation rule for that group (see Proposition S3.3), as well as intricate indirect effects. However, an envelope theorem argument implies that we can bypass the analysis of these complex effects and instead focus on the effect of the score distribution on welfare while holding the allocation rule fixed. Even then, the effect remains difficult to analyze because the surplus function $w_i(s) - \hat{w}_i$ takes both positive and negative values on S_i. Nevertheless, an argument based on the analysis of cumulative surplus functions and the differential form of the objective function shows that improving the score distribution in the first-order stochastic dominance order increases the designer's payoff.

An externality interpretation of comparative statics. This comparative statics result can be interpreted in terms of externalities. We use gaming ability to parameterize the least-cost function. Recall that only the least-cost agents shape (FPC) and thereby the optimal rule. Their presence imposes an externality on other agents. Reducing gaming ability can be interpreted as a removal of least-cost agents, and the resulting impact on remaining agents reflects the externality exerted by these least-cost agents. For example, when gaming ability is low, Proposition 2 implies that least-cost agents exert a positive externality on low-score agents but a negative one on high-score agents.

Comparative statics: the consequences of indirect effects. Comparative statics in the general problem can have rich effects. There can be numerous scenarios depending on initial conditions regarding gaming abilities and score distributions, the number of groups, quotas, and the mass of objects to be allocated. Instead of listing formal results that exhaust all these cases, we find it more effective to focus on an interesting example. We choose an example that illustrates how a quota can be used to shield a group from the indirect effects of transformations occurring in other groups.

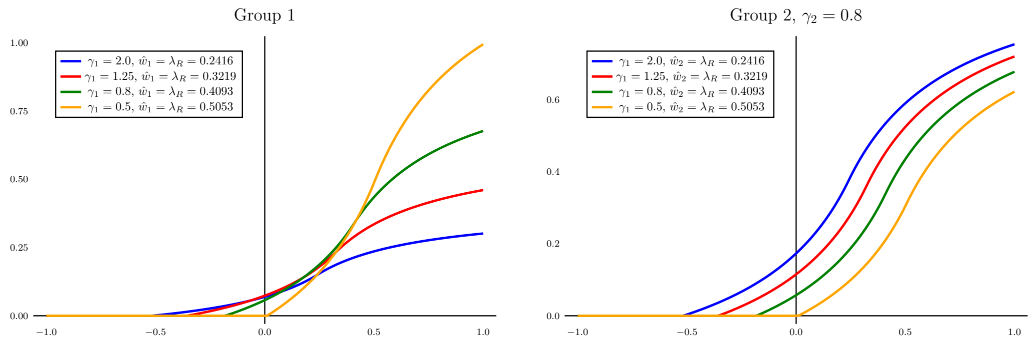

Figure 1: *Group 1's lower gaming ability hurts a low quota group ($\phi_2 = 0.2$). Cost $\mathcal{C}^{\gamma_i}(x) = (1/\gamma_i)x/(1+x)$, score distribution $U(-1,1)$, $\phi_2 = 0.2$, $\gamma_2 = 0.8$, $\bar{\rho} = 0.2$, $\mu_1 = \mu_2 = 0.5$*

Figure 1 and Figure 2 illustrate our example. We consider two equally sized groups, $\mu_1 = \mu_2 = 0.5$, and gradually lower the gaming ability of group 1 from $\gamma_1 = 2$ to $\gamma_1 = 0.5$ while keeping group 2's gaming ability fixed at $\gamma_2 = 0.8$, all corresponding to a slack regime. Both groups are otherwise identical, with scores uniformly distributed on $[-1, 1]$, and a Euclidean cost function $\mathcal{C}^{\gamma}(x) = \frac{1}{\gamma}\frac{x}{(1+x)}$. The mass of objects is $\bar{\rho} = 0.2$ which causes the resource constraint to bind for all parameters considered. There is a quota ϕ_2 of objects reserved for group 2. We consider two scenarios: in the

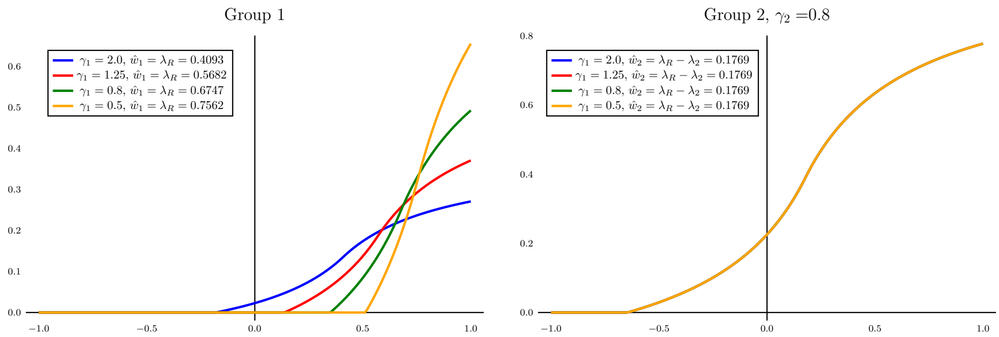

Figure 2: *Group 1's lower gaming ability leaves a high quota group unaffected*

low-quota scenario, the quota $\phi_2 = 0.2$ is never binding, whereas in the high-quota scenario, the quota $\phi_2 = 0.8$ is always binding.

In each case, the decrease in gaming ability of group 1 generates a direct effect that only affects group 1, and an indirect effect that may affect both groups. For each successive decrease in γ_1, since we remain under the slack regime (see Proposition 2), the direct effect on group 1 is to increase the allocation probability for all scores, increasing the mass of objects allocated to this group.

Consider the low-quota scenario, illustrated in Figure 1. As γ_1 decreases, the direct effect results in a higher mass of objects being allocated to group 1. However, when keeping the outside option $\hat{w}_1$ at its original value, the direct effect of each decrease in γ_1 leads to a violation of the resource constraint. To compensate for this direct effect and restore feasibility, the endogenous outside option, which is common to both groups, adjusts upward, generating the indirect effect. This indirect effect shifts group 2's original allocation rule to the right, reducing the probability of receiving an object for all scores. The final allocation rule for group 1 results from the combination of the upward rotation due to the direct effect and the rightward shift due to the indirect effect. It implies that high-score agents gain while low-score agents lose. The overall effect, however, is to increase the mass of objects allocated to group 1 since both groups bear the allocative costs of the indirect effect.

In the high-quota scenario, illustrated in Figure 2, the quota for group 2 is so high that it always binds. To satisfy this quota and ensure that the mass of objects assigned to group 2 remains $0.8\bar{\rho}$, the outside option of group 2 must remain constant. The direct effect of decreasing γ_1 remains the same as in the first scenario. The direct effect of each decrease in γ_1 still leads to a violation of the resource constraint. To compensate for this direct effect while maintaining the quota for group 2, group 1 must bear the full allocative cost of the adjustment through an increase in its outside option. This implies that the overall mass of objects allocated to group 1 remains constant, while high-score agents still gain and low-score agents still lose.

S3 Comparative statics and shape

In this section, we provide additional comparative statics results for the optimal falsification-proof allocation rule of the paper.

S3.1 Shape under shifted linear costs

Proposition S3.1 (Baseline rules under shifted linear costs)**.** *In the shifted linear cost model, the baseline rule is concave in the* (UDD) *case. In the* (UID) *case, it is convex below the eligibility threshold if* $\frac{\kappa''(s)}{\kappa'(s)} \leq \frac{2}{\bar{s}-s}$*, and always concave above the eligibility threshold.*

Proof. Let α denote the baseline rule. For the shifted linear cost family, we have $c_t(t|s) = \kappa(s)$, $c_s(t|s) = -\kappa(s)+(t-s)\kappa'(s)$ and $c_{ts}(t|s) = \kappa'(s)$. If $\kappa'(s) \leq 0$, we are in the (UDD) case. On the growth interval, $\alpha'(s) = c_t(s|s) = \kappa(s)$ and $\alpha''(s) = \kappa'(s) \leq 0$ so the baseline rule is concave along the growth interval.

If $\kappa'(s) \geq 0$, we are in the (UID) case. For ineligible scores on the growth interval,

$$\alpha'(s) = -c_s(m(s)|s) = \kappa(s) - (m(s) - s)\kappa'(s),$$

and

$$\alpha''(s) = (2 - m'(s))\kappa'(s) - (m(s) - s)\kappa''(s).$$

Note that, for the linear shifted cost to satisfy our basic assumptions on costs, we must have $c_s(t|s) = -\kappa(s) + (t - s)\kappa'(s) < 0$ for every t, that is, $\frac{\kappa'(s)}{\kappa(s)} \leq \frac{1}{\bar{s}-s}$. Since $m'(s) < 0$, and $\kappa'(s) \geq 0$, we have $\alpha''(s) \geq 2\kappa'(s) - (m(s) - s)\kappa''(s)$. Then $\alpha''(s) \geq 0$, whenever

$$\frac{\kappa''(s)}{\kappa'(s)} \leq \frac{2}{\bar{s} - s}.$$

For eligible scores, $\alpha'(s) = c_t\big(s|m^{-1}(s)\big) = \kappa(m^{-1}(s))$ and

$$\alpha''(s) = \frac{1}{m'\big(m^{-1}(s)\big)}\kappa'(m^{-1}(s)) \leq 0$$

since m is decreasing. □

S3.2 Returns to scale in falsification

We investigate how the shape of the falsification cost function influences optimal allocations. Specifically, we focus on the class $\mathcal{E}$ of Euclidean cost functions, $c(t|s) = \mathcal{C}\big(|t-s|\big)$, where $\mathcal{C}$ is either concave or convex, and examine the effect of increasing the convexity of the cost function. Intuitively, greater convexity captures lower economies of scale in the magnitude of falsification.

To compare cost functions, we need some normalization. We normalize costs so that the maximum amount of falsification an agent is willing to undertake to get the good is identical for all cost functions and less than $s^*(0)-s_*(0)$. That is, we consider two cost functions $\mathcal{C}, \hat{\mathcal{C}}$ such that $\mathcal{C}^{-1}(1) = \hat{\mathcal{C}}^{-1}(1) < s^*(0) - s_*(0)$. We say that $\hat{\mathcal{C}}$ is more convex than $\mathcal{C}$, and denote $\hat{\mathcal{C}} \succeq_{vex} \mathcal{C}$, if either $\hat{\mathcal{C}}$ is convex and $\mathcal{C}$ is concave, or both are concave and $\mathcal{C}$ is more concave than $\hat{\mathcal{C}}$ in the usual sense, or both are convex and $\hat{\mathcal{C}}$ is more convex than $\mathcal{C}$ in the usual sense.[3] We denote the corresponding optimal falsification-proof allocation rules by $\alpha^*, \hat{\alpha}^*$, and their growth intervals by $I^*, \hat{I}^*$.

Proposition S3.2 (Effect of lowering economies of scale)**.** *If $\hat{\mathcal{C}}$ is more convex than $\mathcal{C}$, then:*

(i) $I^* \subseteq \hat{I}^* \subseteq [s_*(0), s^*(0)]$. Furthermore, $I^* = \hat{I}^* \subset [s_*(0), s^*(0)]$ if both cost functions are concave.

(ii) If $I^* \subset [s_*(0), s^*(0)]$, there exists a threshold $\tilde{s} \in I^*$ such that $\hat{\alpha}^*(s) \leq \alpha^*(s)$ for $s > \tilde{s}$, and $\hat{\alpha}^*(s) \geq \alpha^*(s)$ for $s < \tilde{s}$.

(iii) If $I^* = \hat{I}^* = [s_*(0), s^*(0)]$, then both cost functions are convex and $\hat{\alpha}^*(s) \leq \alpha^*(s)$ for all s if the group has low-score priority, but $\hat{\alpha}^*(s) \geq \alpha^*(s)$ for all s if the group has high-score priority.

Proof. When $\mathcal{C}$ is concave, the growth interval is determined by the equation $m(s_*) - s_* = L$, and does not vary with the cost function, and it is a subset of $[s_*(0), s^*(0)]$ since we assumed $L < s^*(0) - s_*(0)$. For convex cost functions, the growth interval is given by the equation $m(s_*) - s_* = \min\{s^*(0) - s_*(0), 1/\mathcal{C}'(0)\}$ by Proposition 1. Furthermore, if both cost functions are convex, the convex ordering and our normalization imply $\hat{\mathcal{C}}'(0) \leq \mathcal{C}'(0)$, hence $I^* \subseteq \hat{I}^*$.

Next, let $\delta(s) = \hat{\alpha}^*(s) - \alpha^*(s)$. Suppose first that both cost functions are concave, and let $I^* = [s_*, s^*]$ be their common growth interval. In particular, $\delta(s_*) = \delta(s^*) = 0$.

[3] That is, there exists an increasing and concave function $g : [0,1] \to [0,1]$ such that $\mathcal{C} = g \circ \hat{\mathcal{C}}$ when both are concave, or an increasing and convex function $h : [0,1] \to [0,1]$ such that $\hat{\mathcal{C}} = h \circ \mathcal{C}$ if both are convex.

Furthermore, δ is differentiable and

$$\delta'(s) = \begin{cases} \{1 - g' \circ \hat{\mathcal{C}}\big(m(s) - s\big)\}\hat{\mathcal{C}}'\big(m(s) - s\big) & \text{if } s \in [s_*, \hat{s}], \\ \{1 - g' \circ \hat{\mathcal{C}}\big(s - m^{-1}(s)\big)\}\hat{\mathcal{C}}'\big(s - m^{-1}(s)\big) & \text{if } s \in [\hat{s}, s^*] \end{cases},$$

where g is an increasing and concave bijection of $[0, 1]$ such that $\mathcal{C} = g \circ \hat{\mathcal{C}}$. As such, $g'(0) \geq 1 \geq g'(1)$ and g' is a non-increasing function. Since $\mathcal{C}' \geq 0$, this implies δ' is single crossing from the positives to the negatives on $[s_*, \hat{s}]$ and from the negatives to the positives on $[\hat{s}, s^*]$. Therefore, there exists a single threshold $\tilde{s} \in [s_*, s^*]$ such that $\delta(s) \geq 0$ for $s \leq \tilde{s}$ and $\delta(s) \leq 0$ for $s \geq \tilde{s}$.

If the two cost functions are convex, then for $\hat{\mathcal{C}}$ to be more convex than $\mathcal{C}$, it must be that $\hat{\mathcal{C}}'(0) \leq \mathcal{C}'(0)$, which implies (iii).

Let $\tilde{\mathcal{C}}$ be the unique linear cost function that belongs to our normalized class of functions. Since $\tilde{\mathcal{C}}$ is both concave and convex, point (ii) is satisfied when comparing $\tilde{\mathcal{C}}$ to a concave cost function $\mathcal{C}$, and also when comparing a convex cost function $\hat{\mathcal{C}}$ to $\tilde{\mathcal{C}}$. Since $\hat{\alpha}^* - \alpha^* = \hat{\alpha}^* - \tilde{\alpha}^* + \tilde{\alpha}^* - \alpha^*$, it is also satisfied when comparing $\hat{\mathcal{C}}$ to $\mathcal{C}$.

If $I^* = \hat{I}^* = [s_*(0), s^*(0)]$, then both functions must be convex by (i). If the group has low-score priority, then $\alpha^*\big(s_*(0)\big) = \hat{\alpha}^*\big(s_*(0)\big) = 0$, and both allocation rules are linear with respective slopes $\mathcal{C}'(0) \geq \hat{\mathcal{C}}'(0)$, implying $\hat{\alpha}^*(s) \leq \alpha^*(s)$ for all s. If instead the group has high-score priority, the slopes compare in the same way, but the allocation rules are tied at $s^*(0)$ instead of $s_*(0)$, implying $\hat{\alpha}^*(s) \geq \alpha^*(s)$ for all s. □

Lower economies of scale, like higher gaming ability, benefit low-score agents and hurt high-score agents. However, if the diseconomies of scale become too strong, as in case (iii), the effect is uniform across all scores, resembling the impact of high gaming abilities as described in Proposition 2.

S3.3 Score distribution

We examine what changes in the score distribution result in a uniformly higher optimal allocation probability. To simplify notation, we write s for the surplus, so $s = w(s) - \hat{w}$ which effectively transforms the score distribution F. Consequently, the eligibility threshold is fixed at 0. We consider two atomless score distributions, $\hat{F}$ and $\tilde{F}$, whose common support $[\underline{s}, \overline{s}]$ includes a neighborhood of 0. The function $\Delta(s) = \tilde{F}(s) - \hat{F}(s)$ denotes the change in the score distribution.

All distributional effects on the allocation rule are transmitted through the matching functions $\hat{m}(s)$ and $\tilde{m}(s)$. We begin by showing that the allocation rules satisfy $\tilde{\alpha}(s) \geq \hat{\alpha}(s)$ for every s, and for every cost function that meets the conditions in (UID) or (UDD), if and only if $\tilde{m}(s) \leq \hat{m}(s)$ for every $s \leq 0$. We then provide a necessary and sufficient condition on the distributions for the matching function to decrease.

Note that the matching function was defined on the interval $[s_*(0), 0]$, but the lower bound $s_*(0)$ may now depend on the specific score distribution used. To ease the exposition, we extend each matching function $\hat{m}(s)$ and $\tilde{m}(s)$ to the left by setting $\hat{m}(s) = \hat{m}\big(\hat{s}_*(0)\big)$ for $s \leq \hat{s}_*(0)$, and $\tilde{m}(s) = \tilde{m}\big(\tilde{s}_*(0)\big)$ for $s \leq \tilde{s}_*(0)$.

Proposition S3.3 (Effect of score distribution)**.** *The following statements are equivalent:*

(a) The allocation rules satisfy $\tilde{\alpha}(s) \geq \hat{\alpha}(s)$ *for every* s*, and every cost function that satisfies* (UID) *or* (UDD).

(b) The matching functions satisfy $\tilde{m}(s) \leq \hat{m}(s)$ *for every* $s \leq 0$.

(c) For every $0 > s \geq \max\{\hat{s}_*(0), \tilde{s}_*(0)\}$,

$$\int_s^{\tilde{m}(s)} x d\tilde{F}(x) \geq \int_s^{\tilde{m}(s)} x d\hat{F}(x).$$

(d) For every $0 > s \geq \max\{\hat{s}_*(0), \tilde{s}_*(0)\}$,

$$\int_s^0 \{\Delta(s) - \Delta(x)\}dx + \int_0^{\tilde{m}(s)} \{\Delta\big(\tilde{m}(s)\big) - \Delta(x)\}dx \geq 0.$$

Proof.

- *(b) ⇒ (a)*. Suppose (b) holds.

  ◦ *We start by showing (i)* $\tilde{s}_*(0) \leq \hat{s}_*(0)$ *and (ii)* $\tilde{s}^*(0) \leq \hat{s}^*(0)$.

First suppose $\tilde{s}^*(0) = \overline{s}$. Then (ii) must hold, and (i) also because, otherwise, we would have the following contradiction

$$\overline{s} = \tilde{s}^*(0) = \tilde{m}\big(\tilde{s}_*(0)\big) \leq \hat{m}\big(\tilde{s}_*(0)\big) < \hat{m}\big(\hat{s}_*(0)\big) = \hat{s}^*(0),$$

where the first inequality is by (b), and the second inequality because $\hat{m}$ is decreasing on $[\hat{s}_*(0), 0]$.

Next, suppose $\hat{s}_*(0) = \underline{s}$. Then (i) must hold, and (ii) also because, otherwise, we would have the following contradiction

$$\tilde{s}_*(0) = \tilde{m}^{-1}\big(\tilde{s}^*(0)\big) < \tilde{m}^{-1}\big(\hat{s}^*(0)\big) \leq \hat{m}^{-1}\big(\hat{s}^*(0)\big) = \hat{s}_*(0) = \underline{s},$$

where the first inequality is because $\tilde{m}^{-1}$ is decreasing on $[0, \tilde{s}^*(0)]$, and the second inequality is by (b).

If neither of these cases hold, by Lemma 1, (iii), we must have $\tilde{s}_*(0) = \underline{s}$ and $\hat{s}^*(0) = \overline{s}$, which imply (i) and (ii).

An implication of (i) and (ii) is (iii): if $\hat{F}$ has high-score priority, then so does $\tilde{F}$, and if $\tilde{F}$ has low-score priority, then so does $\hat{F}$.

∘ *Next, consider a cost function that satisfies* (UDD)*.*

(b) implies $\tilde{m}(\hat{s}_*) \leq \hat{m}(\hat{s}_*) = \hat{s}^*$, therefore

$$\int_{\hat{s}_*}^{\tilde{m}(\hat{s}_*)} c_{t^+}(x|x)dx \leq \int_{\hat{s}_*}^{\hat{s}^*} c_{t^+}(x|x)dx = 1.$$

Then, using (B) and point (i) we just proved, we must have $\tilde{s}_* \leq \hat{s}_*$.

Then, for every $s \in [\hat{s}_*, \tilde{s}^*]$,

$$\tilde{\alpha}(s) - \hat{\alpha}(s) = \tilde{\Gamma}_{udd}\, \mathbb{1}_{\mathcal{E}} + \int_{\tilde{s}_*}^{\hat{s}_*} c_{t^+}(x|x)dx \geq 0,$$

where $\mathcal{E}$ is the event in which only $\tilde{F}$ has high-score priority (the event in which only $\hat{F}$ has high-score priority is impossible by (iii)). This also implies $\tilde{s}^* \leq \hat{s}^*$, so, for any $s \geq \tilde{s}^*$, we also have $1 = \tilde{\alpha}(s) \geq \hat{\alpha}(s)$. Finally, for $s \leq \hat{s}_*$, we have $\tilde{\alpha}(s) \geq \hat{\alpha}(s) = 0$.

∘ *Finally, consider a cost function that satisfies* (UID)*.*

(b) implies $\tilde{m}(\hat{s}_*) \leq \hat{m}(\hat{s}_*) = \hat{s}^*$, therefore

$$c\big(\tilde{m}(\hat{s}_*)|\hat{s}_*\big) \leq c\big(\hat{s}^*|\hat{s}_*\big) \leq 1.$$

Then, using (B) and point (i) we just proved, we must have $\tilde{s}_* \leq \hat{s}_*$.

(b) also implies $\hat{m}^{-1}(\tilde{s}^*) \geq \tilde{m}^{-1}(\tilde{s}^*) = \tilde{s}_*$, therefore

$$c\big(\tilde{s}^*|\hat{m}^{-1}(\tilde{s}^*)\big) \leq c(\tilde{s}^*|\tilde{s}_*) \leq 1.$$

Then, using (B) and point (ii) we just proved, we must have $\tilde{s}^* \leq \hat{s}^*$.

Then, for every $s \in [\hat{s}_*, 0]$,

$$\tilde{\alpha}(s) - \hat{\alpha}(s) = \tilde{\Gamma}_{uid}\, \mathbb{1}_{\mathcal{E}} - \int_{\tilde{s}_*}^{\hat{s}_*} c_s\big(\tilde{m}(x)|x\big)dx - \int_{\hat{s}_*}^{s} \big\{c_s\big(\tilde{m}(x)|x\big) - c_s\big(\hat{m}(x)|x\big)\big\}dx$$
$$\geq 0,$$

where $\tilde{\Gamma}_{uid} \geq 0$ by definition, the second term is nonnegative by cost monotonicity, and the third term is nonnegative by (UID) and (b).

And for every $s \in [0, \tilde{s}^*]$,

$$\begin{aligned}\tilde{\alpha}(s) - \hat{\alpha}(s) &= \hat{\Gamma}_{uid}\, \mathbb{1}_{\mathcal{E}'} + \int_{\tilde{s}^*}^{\hat{s}^*} c_t\big(x|\hat{m}^{-1}(x)\big)dx \\ &\quad + \int_{s}^{\tilde{s}^*} \big\{c_t\big(x|\hat{m}^{-1}(x)\big) - c_t\big(x|\tilde{m}^{-1}(x)\big)\big\}dx \\ &\geq 0,\end{aligned}$$

where $\hat{\Gamma}_{uid} \geq 0$ by definition, $\mathcal{E}'$ is the event in which only $\hat{F}$ has low-score priority, the second term is nonnegative by cost monotonicity, and the third term is nonnegative by (UID) and (b).

• *(a) ⇒ (b)*. Suppose (a) holds, and consider the family of linear cost functions $c(t|s) = \beta|t - s|$, for $\beta > 0$. By choosing β sufficiently low, we can ensure neither of the allocation rules saturates the probability constraint. In this case, $\hat{\alpha}_\beta(\underline{s}) > 0$ if and only if $\hat{F}$ has high-score priority, but then (a) implies $\tilde{F}$ must have high-score priority as well. Similarly $\tilde{\alpha}_\beta(\overline{s}) < 1$ if and only if $\tilde{F}$ has low-score priority, and then (a) implies $\hat{F}$ has low-score priority as well.

Then $\tilde{\alpha}_\beta(s) = \beta(s - \tilde{s}_*)$ on $[\tilde{s}_*, \tilde{s}^*]$, and $\hat{\alpha}_\beta(s) = \beta(s - \hat{s}_*)$ on $[\hat{s}_*, \hat{s}^*]$. By varying β from 0 to infinity, we have $\tilde{s}_*$ span $[\tilde{s}_*(0), 0)$, and $\hat{s}_*$ span $[\hat{s}_*(0), 0)$. For β sufficiently large, we have both $\tilde{s}_* > \tilde{s}_*(0)$ and $\hat{s}_* > \hat{s}_*(0)$. Pick such a value of β, then by (a), we have

$$-\beta\tilde{s}_* = \tilde{\alpha}_\beta(0) \geq \hat{\alpha}_\beta(0) = -\beta\hat{s}_*,$$

so $\tilde{s}_* \leq \hat{s}_*$. Furthermore, for such a value of β, we must have

$$\tilde{m}(\tilde{s}_*) = \frac{1}{\beta} + \tilde{s}_* \leq \frac{1}{\beta} + \hat{s}_* = \hat{m}(\hat{s}_*) \leq \hat{m}(\tilde{s}_*).$$

Varying β so $\tilde{s}_*$ spans $[s_*(0), 0)$, this shows (b).

• *(b) ⇔ (c) ⇔ (d)*. Since, for all $s < 0$, every x between $\tilde{m}(s)$ and $\hat{m}(s)$ is nonnegative, $\tilde{m}(s) \leq \hat{m}(s)$ is equivalent to $\int_{\tilde{m}(s)}^{\hat{m}(s)} x d\hat{F}(x) \geq 0$. By definition of the matching functions,

$$\int_{s}^{\tilde{m}(s)} x d\tilde{F}(x) = \int_{s}^{\hat{m}(s)} x d\hat{F}(x) = 0,$$

therefore

$$\int_{\tilde{m}(s)}^{\hat{m}(s)} x d\hat{F}(x) - \int_{s}^{\tilde{m}(s)} x d\tilde{F}(x) = \int_{s}^{\tilde{m}(s)} x d\tilde{F}(x) - \int_{s}^{\tilde{m}(s)} x d\hat{F}(x).$$

This shows the equivalence between (b) and (c). The inequality in (d) results from applying integration by parts to (c). □

Hence, a simple first-order stochastic dominance shift is not sufficient to increase the allocation probability for all scores. Since it is challenging to interpret conditions (c) and (d), we provide a more easily interpretable sufficient condition on Δ. We say that Δ *divests* an interval $I \subseteq S$ if every score in I (formally, every measurable subset of I) is less likely under $\tilde{F}$ than under $\hat{F}$. In other words, for every $[s, s'] \subseteq I$,

$$\Delta(s') - \Delta(s) = \{\tilde{F}(s') - \tilde{F}(s)\} - \{\hat{F}(s') - \hat{F}(s)\} \leq 0,$$

or equivalently, if Δ is nonincreasing on I. Conversely, if Δ is nondecreasing on I, we say it *invests* I.

Proposition S3.4. *Suppose there exists $a \in [\underline{s}, 0)$ and $b \in (0, \overline{s}]$ such that*

1. *$\Delta(a) = \Delta(b) = 0$, $\Delta(s) \geq 0$ for all $s \leq a$, and all $s \geq b$;*
2. *Δ divests $[a, 0]$ and invests $[0, b]$;*
3. *$\int_{\underline{s}}^{0} \Delta(x)dx \leq 0$ and $\int_{0}^{\overline{s}} \Delta(x)dx \leq 0$.*

Then the allocation rules satisfy $\tilde{\alpha}(s) \geq \hat{\alpha}(s)$ for every s.

In particular, a change in the distribution that shifts mass from ineligible scores to eligible ones satisfies the conditions of Proposition S3.4, thereby uniformly increasing the allocation probability.

Proof of Proposition S3.4. We show the conditions of the proposition imply that, for every $z < 0 < y$, $\int_{z}^{0}\{\Delta(z) - \Delta(x)\}dx \geq 0$, and $\int_{0}^{y}\{\Delta(y) - \Delta(x)\}dx \geq 0$, which implies condition (d) of Proposition S3.3.

If $z < a$, then

$$\int_{z}^{0} \{\Delta(z) - \Delta(x)\}dx = -z\Delta(z) - \int_{z}^{0} \Delta(x)dx \geq -z\Delta(z) - \int_{\underline{s}}^{0} \Delta(x)dx \geq 0,$$

where the first inequality is from condition 1, and the second inequality is from condition 3 and condition 1, as $\Delta(z) \geq 0$. If $z \geq a$, then condition 2 implies $\Delta(z) \geq \Delta(x)$ for every $x \in [z, 0]$. The proof is symmetric for the integral from 0 to y. □

S4 A continuum as a single-agent

In this appendix, we explain why treating the continuum as a single agent is without loss of generality. The continuum of agents is interpreted as a limit case where the size of the population becomes arbitrarily large. We already discussed why there is no loss of generality in considering allocation rules that only depend on the observed score profile and group identity. In the finite population case, an agent j in group i then receives the good with ex post allocation probability $\alpha_{i,j}(s_j, s_{-j})$. As often in mechanism design, the problem can be reformulated as one of choosing interim allocation probabilities $\alpha_{i,j}(s_j) = \mathbb{E}_{s_{-j}}\, \alpha_{i,j}(s_j, s_{-j})$. Furthermore, given the symmetry of our setup, we can assume symmetry across agents of the same group, so we can write $\alpha_i(s)$ for the interim allocation probability for an agent with score s in group i. Then the *interim problem* of optimizing over symmetric interim allocation rules in any finite population is exactly the program we solve in the continuum.[4] However, to find a solution to the initial program, we need to ensure that the interim allocation rules that solve the interim program are feasible in the sense that they can be obtained from an ex-post allocation rule. In the finite population case, the exact condition for this to be possible can be derived from Che, Kim and Mierendorff (2013) which generalizes the condition of Border (1991) to setups with multiple goods and quotas. In the limit case of the continuum, however, the interim rules can be used directly as ex post allocation rules that only depend on each agent's score, so feasibility is automatically satisfied.

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